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DYNAMIC SNAP-THROUGH OF AN ELASTIC SHELL SUBJECTED TO A PULSED LOAD*

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Dynamic instability "in the large" is investigated for an elastic shallow shell subjected to a pulsed load (PL) and a more general non-linear elastic continuous conservative system with potential energy of the "square of the norm plus a weakly continuous functional" kind with Rayleigh friction and a given initial velocity. An energy approach /1, 2/ developed in /3-6/ is used to analyse the dynamic snap-through (DS) of the non-linear elastic system subjected to a stationary step load. The problem is considered in an exact infinite-dimensional formulation. By using the concept of an equilibrium stability trough and reserve /7/, definitions are given of the dynamic stability of a system, the critical PL of its DS and the astatic critical PL. The latter is determined from the stationary problem and yields the lower bound for those PL values for which DS occurs.

It is established that a necessary condition for the DS of a system subjected to a PL is the existence of a saddle point of the potential energy for the same load-free system at the boundary of a stable zero equilibrium trough. It is proved that this necessary condition is satisfied for sufficiently thin strictly convex shells of revolution with movable and fixed hinge-support as well as for a certain class of arbitrary strictly convex shells with a movable hinge support.

Therefore, the stability reserve has a graphic mechanical meaning as an exact upper bound of the kinetic energy which can be added to the system at rest so that it does not reach the least saddle points among the energy heights leading from a zero equilibrium trough to troughs of other equilibria.

Results of computer calculations of the critical PL of dynamic snap-through and astatic snap-through for spherical and conical shells are presented. Depending on the boundary conditions, the lower limits are determined for the ratio between the shell rise and its thickness, for which DS is possible under the action of a PL.

Note that the reasoning associated with estimating the kinetic energy needed to overcome the energy barrier in the problem of DS under the action of a PL on a system with two degrees of freedom, obtained by the Bubnov-Galerkin method from the vibrations equations for an elastic arch, were first applied in /1/. The extensions to finite-dimensional models with a large number of degrees of freedom made by different authors are reflected in /2, 8, 9/.

1. On the formulation of the DS problem. In a separable Hilbert space H we consider the vibrations equation of a continuous conservative mechanical system subjected to a PL in the presence of a viscous friction force /10/

$$\omega_{tt} + \beta_0 B\omega_t + I'(\omega) = 0,\ I'(\omega) = \operatorname{grad}_H I(\omega) \tag{1.1}$$

$$I(\omega) = {}^1/_2 \|\omega\|_{H_A}^2 - \varphi(\omega),\ \|\omega\|_{H_A}^2 = (A\omega, \omega)_H$$

$$\omega|_{t=0} = 0,\ \omega_t|_{t=0} = v \in H \tag{1.2}$$

Here $\omega(t)$ is an unknown vector-function of time t, $I(\omega)$ is the potential energy, H_A is the energy space of the selfadjoint positive-definite operator A, $\varphi(\omega)$ is a weakly continuous functional in H_A. The Rayleigh friction is given by a selfadjoint positive-definite operator B and the coefficient of friction β_0. The functional $I(\omega)$ is given in the whole space H_A and is growing in ω. The work of the external forces does not occur in $I(\omega)$ since the PL is reduced until the system starts to move and therefore communicates a field of velocities v to a system at rest. The properties of the functional $I(\omega)$ are established in /6/.

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Let the point $\omega_0 = 0 \in H_A$ correspond to zero stable equilibrium of the system and let $J(0)$ be its trough which is defined, according to /7/, as a connected set in H_A that contains zero and consists of points ω satisfying the condition $I(\omega) < j^*$. Here j^* is the upper limit of those energy values j for which the sets $\{\omega \in H_A : I(\omega) < j\}$ contain no stable equilibria different from zero. By virtue of a theorem in /6/, we obtain that from the ambiguity of the zero equilibrium there follows the existence of at least one saddle point γ on the boundary of the trough $J(0)$. In this case the stability reserve of the zero equilibrium /7/

$$Z(0) = I(\gamma) - I(0) \tag{1.3}$$

has a graphic mechanical meaning as an exact upper limit of the kinetic energy which can be given to the system in the zero position so that it will remain in the trough $J(0)$ for all $t > 0$.

Many problems of DS under the action of a PL are included in the following general scheme. Let the initial velocity $v = v(\alpha)$ depend continuously on the PL parameter α, where $v(0) = 0$. For the value $\alpha_0 = 0$ let the system be in the zero-th stable equilibrium position and at the time $t = 0$ let the parameter α change abruptly from α_0 to α°. The question arises as to whether the system remains in the trough $J(0)$ or leaves it as time passes. Therefore, the situation reduces to investigating the behaviour of the solution of the Cauchy problem for system (1.1) with the initial data $\omega(0) = 0$, $\omega_t(0) = v(\alpha)$. If $\omega(t) \in J(0)$ for all $t \geqslant 0$, then we will say that there is no DS. If $\omega(t)$ for a certain t_* turns out to be outside the trough $J(0)$, then DS occurs. Evidently DS is not possible if $\omega_0 = 0$ is the only equilibrium of system (1.1).

2. Critical PL.

The domain of possible motions M_0 of system (1.1) for $\beta_0 = 0$ is determined by the inequality

$$\tfrac{1}{2}\| \omega_t \|_H^2 + I(\omega) \leqslant \tfrac{1}{2}\| v \|_H^2 + I(0) \tag{2.1}$$

For $\alpha = \alpha^\circ$ let the domain $M_0(\alpha^\circ)$ satisfy the condition $M_0(\alpha^\circ) \subset J(0)$. Then there is no DS. Let us increase α starting with α°. DS becomes possible for the least value of α for which the saddle point in $\partial J(0)$ falls in the domain $M_0(\alpha)$.

We determine the critical PL of the DS α_d^* for the fundamental equilibrium $\omega_0 = 0$ by setting it equal to the upper bound of those values of α for which the motion of system (1.1) and (1.2) remains in the trough $J(0)$ for all $t > 0$. Therefore, when α exceeds α_d^* slightly, the motion mentioned will emerge from the trough of the equilibrium $\omega_0 = 0$ at a certain time, according to the definition this will indeed be a DS.

The calculation of α_d^* involves integration of the non-stationary system (1.1) for different α in a time segment not determined in advance, which requires a considerable amount of computer time.

Furthermore, we introduce the astatic critical PL α_a of the equilibrium $\omega_0 = 0$ as the least solution of the equation

$$I(\omega_a) = \tfrac{1}{2}\| v(\alpha) \|_H^2 + I(0) \tag{2.2}$$

where ω_a is an unstable equilibrium on $\partial J(0)$ so that $I'(\omega_a) = 0$.

For values of $\alpha = \alpha_a$ and α slightly exceeding α_a, the motion directed along the connecting points $\omega_0 = 0$ and $\omega_a \in \partial J(0)$ which is a geodesic and has the initial velocity field with norm $\| v(\alpha)\|_H$, leaves the system from $J(0)$. Obviously, $\alpha_a \leqslant \alpha_d^*$ so that the astatic critical PL sets a lower limit for the dynamic load. It can be assumed that it is more informative from the practical viewpoint since it is referred to the more general initial conditions (instead of the condition $\omega_t(0) = v(\alpha)$ only the energy equality $\| \omega_t(0)\|_H = \| v(\alpha)\|_H$ is required). Meanwhile, the quantity α_a is considerably easier to calculate in many cases since only stationary problems need be considered for this.

3. Non-stiffness and DS of elastic shells.

The vibrations equations of an elastic shallow shell belong to the class of equations of the form (1.1) under consideration /6, 10/, and consequently, all our reasoning holds for them. It follows from Sect.2 that the necessary condition for a DS of a system subjected to a PL is the existence of a potential energy saddle point on the stable zero-th equilibrium trough boundary for the same system without a load. This condition is known to be violated for stiff mechanical systems for which there is only the zero-th equilibrium for no load in conformity with the definition /11/. For stiff systems DS is impossible under the action of a PL for any α.

We will examine the problem of DS in more detail for elastic shallow shells. It has been shown /11/ that plates of arbitrary shape with natural homogeneous boundary conditions, as well as shells with a thickness having a definite lower limit, are stiff. As the relative thin-walledness diminishes, a shell can lose it stiffness, i.e., have different equilibrium modes from the unstressed one when there is no load. It is well-known /11-13/ that sufficiently thin strictly convex shells with a hinge support lack stiffness, and in order to establish the possibility of DS under the action of a PL for them it is necessary to prove the existence of at least one potential energy saddle point of such a shell when there is no load.

We will write the system of equilibrium equations of a thin elastic strictly convex unstressed shell S with moving hinge support in the form

$$\varepsilon^2\Delta^2F + {}^1/_2\,[w, w] - [z, w] = 0 \tag{3.1}$$
$$\varepsilon^2\Delta^2w - [w - z, F] = 0,\ z\,|_\Gamma = 0$$
$$[w, F] = w_{xx}F_{yy} + w_{yy}F_{xx} - 2w_{xy}F_{xy}$$
$$[F = F_\rho = w]_\Gamma = 0,\quad \Gamma_1 w = [w_{\rho\rho} - \nu\varkappa_1 w_\rho]_\Gamma = 0 \tag{3.2}$$

The dimensionless quantities in (3.1) and (3.2) are related by the dimensional formulas /13/

$$\{W, S, x_1, y_1, n, \tau, \varkappa_0^{-1}\} = a\,\{w, z, x, y, \rho, s, \varkappa_1^{-1}\}$$
$$\Phi = Ea^2\varepsilon^2F,\ \varepsilon^2 = h/(a\gamma),\ \gamma^2 = 12\,(1 - \nu^2)$$

Here W is the deflection, Φ is the Airy stress function, x_1, y_1 are rectangular Cartesian coordinates, a is the characteristic dimension of a simply-connected strictly convex domain D which the shell with middle surface S occupies in planform. The small parameter ε^2 is the shell relative thin-walledness, h is the thickness, ν is Poisson's ratio, and $n, \tau, \varkappa_0$ are the internal normal, the arc length, the curvature of the contour Γ bounding the domain D. For any $x, y \in D + \Gamma$ the surface z satisfies the condition

$$z_{xx}m^2 - 2z_{xy}mn + z_{yy}n^2 \leqslant -\beta\,(m^2 + n^2),\ \forall m, n \in R \tag{3.3}$$

where β is a certain positive constant.

Together with the zero-th equilibrium $F = w = 0$ problem (3.1) and (3.2) can have an equilibrium $V_* = (F_*, w_*)$ close to a symmetrically reflected one as $\varepsilon \to 0$ and the following asymptotic expansions $V_\varepsilon = (F_\varepsilon, w_\varepsilon)$ hold for it /13/:

$$F_* \sim F_\varepsilon = \varepsilon^2\,(s_1 + F^\circ),\ w_* \sim w_\varepsilon = 2z + \varepsilon^2\,(s_2 + w^\circ) \tag{3.4}$$
$$\{s_1, s_2\} = \sum_{i=1}^{n} \varepsilon^{i-1}\,\{h_i(t), g_i(t)\}\,\psi\left(\frac{\rho}{\delta}\right),\quad t = \frac{\rho}{\varepsilon} \geqslant 0$$
$$F^\circ = \sum_{i=2}^{n} \varepsilon^{i-2}F_i(x, y) + \varepsilon^{n-1}\gamma_3(x, y)$$
$$w^\circ = \sum_{i=2}^{n} \varepsilon^{i-2}w_i(x, y) + \varepsilon^{n-2}\gamma_1(x, y) + \varepsilon^{n-1}\gamma_2(x, y)$$
$$h_1 = -C_1e^{-\lambda t}\sin\left(\lambda t + \frac{\pi}{4}\right),\quad g_1 = -C_1e^{-\lambda t}\sin\left(\lambda t - \frac{\pi}{4}\right)$$
$$C_1 = \sqrt{2}\,[2\nu - \lambda^{-2}z_{\rho\rho}(s)];\ \lambda^2 = {}^1/_2\varkappa_1 z_\rho(s) > 0,\ s \in \Gamma$$

The functions F_i and w_i are obained by direct expansion of the solution in powers of ε as a result of the first iteration process. The boundary layer functions h_i, g_i which are determined in the second iteration process /13-15/ compensate their residual in satisfying the boundary conditions on Γ. The smoothing function $\psi(\eta)$ equal unity for $\eta = \rho\delta^{-1} \leqslant {}^1/_3$ and zero for $\eta \geqslant {}^2/_3$, where δ is the band width in which the internal normals to Γ do not intersect. In order for the vector-function V_ε to satisfy all the boundary conditions exactly on Γ, components of the order of ε^n and ε^{n+1} are appended with arbitrary sufficiently smooth functions $\gamma_1, \gamma_2, \gamma_3$ satisfying the boundary conditions

$$[\gamma_1 = \gamma_2 + g_n = \gamma_3 + h_n = \gamma_{3,\rho}]_\Gamma = 0$$
$$\Gamma_1\,(\gamma_1 + w_n) = \nu\varkappa_1 g_{n,t}\,|_{t=0};\ \Gamma_1\gamma_2 = 0$$

We will consider problem (3.1) and (3.2) as an operator equation $P(V) = 0$. Here $V = (F, w)$ and the operator $P\colon X \to Y$, where the space Y consists of the vector-function $f = (f_1, f_2)$ with the finite norm

$$\|f\|_Y^2 = \|f_1\|^2 + \|f_2\|^2 = \int_D (f_1^2 + f_2^2)\,dx\,dy$$

and the space X is the closure of the sets of smooth vector-functions $V = (F, w)$ with finite norm

$$\|V\|_X^2 = \|F\|^2_{W_2^{(4)}} + \|w\|^2_{W_2^{(4)}}$$

that satisfy (3.2).

Theorem 1. Let $z(s)$ satisfy the condition

$$-[z_{\rho\rho}(\varkappa_1 z_\rho)^{-1}]_\Gamma < {}^1/_2(1-\eta_0)(1+e^{-\pi})^{-1} - \nu \tag{3.5}$$

for any $s \in \Gamma$, where η_0 is an arbitrarily small positive number. Then for $\varepsilon \ll 1$ problem (3.1)-(3.3) has the solution V_* with asymptotic form V_ε, where

$$\max_{D+\Gamma}|F_* - F_\varepsilon| + \max_{D+\Gamma}|w_* - w_\varepsilon| \leqslant m_1\varepsilon^{n+1}, \quad n = 2; 3; \ldots \tag{3.6}$$
$$\max_{D+\Gamma}|D^{(k)}(F_* - F_\varepsilon)| + \max_{D+\Gamma}|D^{(k)}(w_* - w_\varepsilon)| \leqslant m_1\varepsilon^{n+2-k}$$

The solution V_* realizes the local minimum potential energy of the shell.

(Here and everywhere henceforth, m_i and c_i are positive constants independent of ε. The symbol $D^{(k)}$ denotes any partial derivative of order $k = 1, 2.$)

Proof. We consider the system of equations

$$P'_{V_\varepsilon}(V) = f, \; V = (F, w) \in X, \; f = (f_1, f_2) \in Y, \; P'_{V_\varepsilon}(V) \equiv$$
$$(\varepsilon^2\Delta^2 F + [z, w] + \varepsilon^2[s_2 + w^0, w], \; \varepsilon^2\Delta^2 w - [z, F] -$$
$$\varepsilon^2[s_2 + w^0, F] - \varepsilon^2[s_1 + F^0, w])$$

with boundary conditions (3.2). Here P'_{V_ε} is the Frechet derivative of the operator P on the element V_ε.

We multiply the first equation of the system by $F + \varepsilon^\alpha w$ and the second by $w - \varepsilon^\alpha F$. Integrating over the domain D and combining, we obtain

$$\varepsilon^2\|\Delta F\|^2 + \varepsilon^2\|w\|_2^2 + \varepsilon^{2}\nu\int_\Gamma \varkappa_1 w_\rho{}^2\,ds - \varepsilon^{2+\alpha}\int_\Gamma \Delta F w_\rho\,ds - \varepsilon^2(s_1 + F^0, w, w) +$$
$$\varepsilon^\alpha(z, w, w) + \varepsilon^\alpha(z, F, F) + \varepsilon^{2+\alpha}\{(s_2 + w^0, w, w) + (s_1 + F^0, w, F) +$$
$$(s_2 + w^0, F, F)\} = \int_D \{f_1(F + \varepsilon^\alpha w) + f_2(w - \varepsilon^\alpha F)\}\,dx\,dy$$
$$(a, b, c) = \int_D [a, b]\,c\,dx\,dy = \int_D a\,[b, c]\,dx\,dy$$
$$\|w\|_2^2 = \int_D (w_{xx}^2 + w_{yy}^2 + 2w_{xy}^2)\,dx\,dy; \quad \|\cdot\| = \|\cdot\|_{L_2(D)}$$

Applying the reasoning of Lemma 4.3 in /13/, we arrive at the following conclusion. If for $\varepsilon \ll 1$, $1 < \alpha < 2$ and a certain number η_0 $(0 < \eta_0 < 1)$ the inequality

$$(1-\eta_0)\varepsilon^2\Big[\|w\|_2^2 + \nu\int_\Gamma \varkappa_1 w_\rho{}^2\,ds\Big] - \varepsilon^2\int_D (s_1 + F^0)[w, w]\,dx\,dy + \varepsilon^\alpha\beta\|\nabla w\|^2 \geqslant 0 \tag{3.7}$$

holds for any function $w \in W_2^{(4)}$ with the boundary conditions $w|_\Gamma = \Gamma_1 w = 0$, then the following estimate holds:

$$\|[P'_{V_\varepsilon}]^{-1}\|_{(Y\to X)} \leqslant c_3\varepsilon^{-4}$$

By virtue of (3.4) we have $s_1 + F^0 = h_1 + F_2 + 0(\varepsilon)$ and it is sufficient to prove satisfaction of (3.7) with $(s_1 + F^0)$ replaced by $(h_1 + F_2)$. The function F_2 is found from the boundary value problem /13/

$$[z, F_2] = 2\Delta^2 z, \quad F_2|_\Gamma = -h_1|_\Gamma = 2[\nu - z_{\rho\rho}(\varkappa_1 z_\rho)^{-1}]_\Gamma \tag{3.8}$$

We will represent F_2 in the form

$$F_2 = F_{20} + G, \; [z, F_{20}] = 0, \; F_{20}|_\Gamma = -h_1|_\Gamma$$
$$[z, G] = 2\Delta^2 z, \; G|_\Gamma = 0$$

Applying (3.8) and integrating by parts, we obtain

$$|(h_1 + F_{20}, w, w)| \leqslant \max_{D+\Gamma}|h_1 + F_{20}|\,\|w\|_2^2 \tag{3.9}$$
$$|(G, w, w)| = \Big|\int_D (G_{xx}w_y{}^2 + G_{yy}w_x{}^2 - 2G_{xy}w_x w_y)\,dx\,dy\Big| \leqslant$$
$$c_4\|\nabla w\|^2, \quad \max_{D+\Gamma}|h_1 + F_{20}| \leqslant \max_{D+\Gamma}|h_1| + \max_{D+\Gamma}|F_{20}| \leqslant$$
$$2(1 + e^{-\pi})[\nu - z_{\rho\rho}(\varkappa_1 z_\rho)^{-1}]_\Gamma$$

This latter inequality results from the maximum principle for homogeneous elliptic equations and (3.4) and (3.8). Now (3.7) follows from (3.5) and (3.9). The existence of the solution V_* with the asymptotic form V_ε and the estimate (3.6) for the remainder term is obtained from Theorem 4.1 in /13/ and the estimates

$$\| P(V_\varepsilon) \|_Y \leqslant c_1 \varepsilon^n, \quad \| P_V{}' \|_{(X \to (X \to Y))} \leqslant c_3$$

(here the misprints made in (2.11), (4.9), (4.11), and (4.14) in /13/ have been corrected).

Let us prove the stability of V_*. The solution of the boundary value problem (3.1) and (3.2) is the critical point of the functional

$$I_\varepsilon(w) = \frac{\varepsilon^2}{2} \int_D \{(\Delta w)^2 - (1-\nu)[w, w] + (\Delta F)^2\}\, dx\, dy \tag{3.10}$$

proportional to the shell potential energy when there is no load. The function F is determined uniquely from the solution of the first equation in (3.2) for a given $w \in W_2^{(2)}$ with the boundary conditions $w|_\Gamma = \Gamma_1 w = 0$.

Evaluating the second variation of the functional in the solution $V_* = (F_*, w_*)$, we have /16/

$$\delta^2 I_\varepsilon(w_*) = \frac{\varepsilon^2}{2}\left(\|\eta\|_2^2 + \nu \int_\Gamma \varkappa_1 \eta_\rho^2\, ds\right) + \varepsilon^2 \|\Delta\varphi\|^2 + (F_*, \eta, \eta) \tag{3.11}$$

Here η is an allowable variation of w, and φ is an allowable variation of F and is determined from the boundary value problem

$$\varepsilon^2 \Delta^2 \varphi + [w_* - z, \eta] = 0, \quad \varphi|_\Gamma = \varphi_\rho|_\Gamma = 0 \tag{3.12}$$

Multiplying (3.12) by $\varepsilon^\alpha \eta$ and integrating over the domain D, we obtain, taking (3.4) into account,

$$\begin{gathered}\varepsilon^{2+\alpha}\left\{\int_D \Delta\varphi\Delta\eta\, dx\, dy + \int_\gamma \Delta\varphi\eta_\rho\, ds + (s_2 + w^\circ, \eta, \eta)\right\} - \\ \varepsilon^\alpha \int_D (z_{xx}\eta_y^2 + z_{yy}\eta_x^2 - 2z_{xy}\eta_x\eta_y)\, dx\, dy = \varepsilon^\alpha (w_\varepsilon - w_*, \eta, \eta)\end{gathered} \tag{3.13}$$

By using the inequalities

$$\varepsilon^{2+\alpha} \int_D \Delta\varphi\Delta\eta\, dx\, dy \leqslant {}^1/_2 \varepsilon^2 (\|\eta\|_2^2 + \|\Delta\varphi\|^2)$$

for $1 < \alpha < 2$, we deduce from (3.11) and (3.13)

$$\begin{gathered}\delta^2 I_\varepsilon(w_*) \geqslant \frac{\varepsilon^2}{2}\left(1 - \frac{2}{3}\eta_0\right)\left(\|\eta\|_2^2 + \nu \int_\Gamma \varkappa_1\eta_\rho^2\, ds + \|\Delta\varphi\|^2\right) + \\ \frac{\varepsilon^2}{2}(s_1 + F^\circ, \eta, \eta) + \frac{1}{2}(F_* - F_\varepsilon, \eta, \eta) + \\ \frac{2}{3}\eta_0\varepsilon^{2+\alpha}\left\{\int_\Gamma \Delta\varphi\eta_\rho\, ds + (s_2 + w^\circ, \eta, \eta)\right\} + \\ \frac{2}{3}\eta_0\varepsilon^\alpha\left\{(w_* - w_\varepsilon, \eta, \eta) - \int_D (z_{xx}\eta_y^2 + z_{yy}\eta_x^2 - 2z_{xy}\eta_x\eta_y)\, dx\, dy\right\}\end{gathered} \tag{3.14}$$

Applying the reasoning performed in deriving (4.18) in /13/, and taking account of the estimate (3.6), we have

$$\begin{gathered}\Delta^2\varphi = f_0, \quad \varphi|_\Gamma = \varphi_\rho|_\Gamma = 0 \\ f_0 = -\varepsilon^{-2}\{[z, \eta] + \varepsilon^2 [s_2 + w^\circ, \eta] + [w_* - w_\varepsilon, \eta]\} \\ \|f_0\| \leqslant m_2\varepsilon^{-2}\|\eta\|_2, \quad \int_D |[\eta, \eta]|\, dx\, dy \leqslant \|\eta\|_2^2 \\ \max_{D+\Gamma} |\varepsilon\Delta\varphi| \leqslant m_3(\|\Delta\varphi\| + \|\eta\|_2) \\ \left|\int_\Gamma \varepsilon\Delta\varphi\eta_\rho\, ds\right| \leqslant c_5\left(\|\eta\|_2^2 + \|\Delta\varphi\|^2 + \nu\int_\Gamma \varkappa_1\eta_\rho^2\, ds\right)\end{gathered} \tag{3.15}$$

Now applying (3.3)-(3.9) and (3.15), and assuming ε to be so small that the inequalities $\varepsilon^2 c_4 < \eta_0\beta\varepsilon^\alpha/3$, $\varepsilon^{\alpha-1}c_5 < {}^1/_6$, $\varepsilon^\alpha m_5 < {}^1/_6$, $m_5 = \max_{D+\Gamma} |s_2 + w^\circ|$ are satisfied for $1 < \alpha < 2$, we obtain from (3.14)

$$\begin{gathered}\delta^2 I_\varepsilon(w_*) \geqslant \frac{\varepsilon^2}{2}(1 - \eta_0)\left(\|\eta\|_2^2 + \nu\int_\Gamma \varkappa_1\eta_\rho^2\, ds + \|\Delta\varphi\|^2\right) + \\ \frac{1}{3}\eta_0\varepsilon^\alpha\beta\|\nabla\eta\|^2 > 0\end{gathered}$$

Therefore, the second variation of the potential energy is a positive-definite mode and by virtue of /16/ the solution V_* with the asymptotic form V_ε corresponds to stable equilibrium.

Theorem 2. For sufficiently small ε problem (3.1)-(3.3), (3.5) together with the stable zero-th equilibrium and stable equilibrium V_* close to the symmetrically reflected one, there is at least one unstable equilibrium corresponding to the saddle-point of the shell potential energy.

Proof. The functional I_ε in (3.10) satisfies the conditions of Sect.1 in /6/. The stability of the zero-th equilibrium for all $\varepsilon > 0$ is easily obtained from (3.11) for $w_* = F_* = 0$ by virtue of /16/. Now, the assertion formulated results from Theorems 2.1 and 2.2 in /6/.

We are interested in the problem of giving a foundation for the asymptotic form without the constraining condition (3.5) on the equation of the surface z. This condition is obviously not related to the substance of the matter and it is understood the asymptotic expansions (3.4) can be used as $\varepsilon \to 0$ to compute the above-mentioned non-trivial equilibrium V_* of an arbitrary shallow strictly convex shell with a moving hinge support.

For an unloaded shell with fixed hinge support the system of equilibrium equations is written in the form (3.1) with the boundary conditions

$$[w = w_{\rho\rho} - \nu\varkappa_1 w_\rho = F_{\rho\rho} + \nu\varkappa_1 F_\rho - \nu F_{ss}]_\Gamma = 0$$
$$\varepsilon^2 [F_{\rho\rho\rho} + 3\varkappa_1 F_{ss} + (2+\nu) F_{\rho ss} + (2+\nu)\varkappa_{1,s} F_s -$$
$$\varkappa_1^2 (1-\nu) F_\rho]_\Gamma - \varkappa_1 \left[w_\rho \left(\frac{1}{2} w_\rho - z_\rho\right)\right]_\Gamma = 0$$

For equilibrium close to the symmetrically reflected equilibrium $V_* = (F_*, w_*)$ of a strictly convex shell, asymptotic representations are constructed here as $\varepsilon \to 0$

$$F_* \sim \varepsilon^2 A_1 \Lambda^{-2} e^{-\Lambda t} \cos \Lambda t, \quad w_* \sim 2z(x, y) + \varepsilon^2 A_1 \lambda^{-2} e^{-\lambda t} \sin \lambda t,$$
$$\lambda^2 = \frac{1}{2} \varkappa_1 z_\rho (s) > 0, \quad s \in \Gamma$$
$$A_1 = [z_{\rho\rho} - \nu\varkappa_1 z_\rho]_\Gamma, \ \varkappa_1 = \varkappa_1 (s) > 0, \ t = \rho\varepsilon^{-1}$$

There is no foundation for the asymptotic form in this case since there is an error in Lemma 1 in /17/.

Assuming axisymmetric strain of shells of revolution, (3.1) reduces to the system

$$\varepsilon^2 A v - {}^1/_2 u^2 + \theta u = 0, \ \varepsilon^2 A u + uv - \theta v = 0 \tag{3.16}$$
$$A(\ ) = -r(\ )'' - (\ )' + r^{-1}(\ ), \ v = F', \ w = u'$$
$$\varepsilon^2 = h/(a\gamma), \ \theta = z' \leqslant -\beta r, \ \beta > 0, \ (\ )' = d(\ )/dr$$

Here a is the radius of the shell reference outline.

Consider system (3.16) together with the boundary conditions

$$1) \ \left|\frac{v}{r}, \frac{u}{r}\right|_{r=0} < \infty, \quad v(1) = 0, \quad Mu = [u' + \nu u]_{r=1} = 0 \tag{3.17}$$
$$2) \ \left|\frac{v}{r}, \frac{u}{r}\right|_{r=0} < \infty, \quad Nv = [v' - \nu v]_{r=1} = 0, \quad Mu = 0$$

that correspond to moving and fixed hinge support.

The asymptotic expansions as $\varepsilon \to 0$ for an equilibrium $V_s = (v_*, u_*)$ close to the symmetrically reflected one are constructed in the form /12, 13/

$$v_* \sim v_\varepsilon = \varepsilon (s_3 + \varepsilon v^\circ), \ u_* \sim u_\varepsilon = \varepsilon (s_4 + \varepsilon u^\circ) + 2\theta \tag{3.18}$$
$$s_3 = \sum_{i=1}^{n+1} \varepsilon^{i-1} [h_i(t) + \alpha_i], \quad s_4 = \sum_{i=1}^{n+1} \varepsilon^{i-1} [g_i(t) + \beta_i]$$
$$v^\circ = \sum_{i=2}^{n} \varepsilon^{i-2} v_i(r) + \varepsilon^{n-1} \gamma_1(r)$$
$$u^\circ = \sum_{i=2}^{n} \varepsilon^{i-2} u_i(r) + \varepsilon^{n-1} \gamma_2(r), \quad V_\varepsilon = (v_\varepsilon, u_\varepsilon)$$

$$1) \ h_1 = 2K \sin bt, \ g_1 = -2K \cos bt, \ t = (1-r)/\varepsilon$$
$$2) \ h_1 = \sqrt{2} K \sin (bt + \pi/4), \ g_1 = \sqrt{2} K \sin (bt - \pi/4)$$
$$K = [d\theta/dr + \nu\theta]_{r=1} b^{-1} e^{-bt}, \ b = \sqrt{-{}^1/_2 \theta(1)}$$

The functions v_i, u_i, h_i, g_i are found as a result of the first and second iteration processes /12-15/, while the functions α_i, β_i compensate the residue of exponential order of smallness in ε for h_i, g_i to zero. In order for the vector function V_ε to satisfy all the boundary conditions exactly, components of the order ε^{n+1} are appended with the arbitrary sufficiently smooth functions γ_1, γ_2 satisfying the boundary conditions

$$\gamma_1(0) = \gamma_2(0) = 0,\ M\gamma_1 = -\nu g_{n+1}|_{t=0}$$
$$1)\ \gamma_2|_{r=1} = -h_{n+1}|_{t=0},\ 2)\ N\gamma_2 = \nu h_{n+1}|_{t=0}$$

Theorem 3. Boundary conditions (3.16) and (3.17) for sufficiently small ε, together with the stable zero-th equilibrium and the stable **equilibrium** close to the symmetrically reflected one, have at least one unstable equilibrium corresonding to the saddle-point of the potential energy.

The existence of a non-trivial solution V_* with asymptotic form V_ε and the estimate $\max |v_* - v_\varepsilon| + \max |u_* - u_\varepsilon| \leqslant m_2\varepsilon\,\delta^{n+1}$ where the maximum is taken for $0 \leqslant r \leqslant 1$, were proved in /12, 13/.

We will prove the stability of the solution V_*. We introduce the Hilbert space H_1, the closure of the sets of doubly continuously differentiable functions in $[0,1]$, that satisfy the boundary conditions for u in (3.17) with a finite norm generated by the scalar product

$$(u_1, u_2)_{H_1} = \int_0^1 \left[r u_1' u_2' + \frac{u_1 u_2}{r} + \nu (u_1 u_2)' \right] dr$$

The solution V_* is the critical point of the functional

$$I_*(u) = \frac{\varepsilon^2}{2} \| u \|^2_{H_1} + \frac{\varepsilon^2}{2} \int_0^1 \left[r v'^2 + \frac{v^2}{r} - 2\nu v v' \right] dr \tag{3.19}$$

proportional to the potential energy of a shell of revolution when there is no load. Here the function v is uniquely defined for a given $u \in H_1$ from the solution of the first equation in (3.16) with appropriate boundary conditions for v in (3.17).

We have for the second variation of the functional in V_*

$$\delta^2 I_*(u_*) = \frac{\varepsilon^2}{2} \| \eta \|^2_{H_1} + \frac{\varepsilon^2}{2} \| \varphi \|^2 - \frac{1}{2}\varepsilon^2 \nu \varphi^2(1) + \frac{1}{2} \int_0^1 v_* \eta^2\, dr, \tag{3.20}$$
$$\| \varphi \|^2 = \int_0^1 \left[r\varphi'^2 + \frac{1}{r}\varphi^2 \right] dr$$

Here η is an allowable variation of the function u and φ is determined from the boundary value problem

$$\varepsilon^2 A\varphi + (\theta - u_*)\eta = 0,\ |\varphi r^{-1}|_{r=0} < \infty$$
$$1)\ \varphi(1) = 0,\ 2)\ [\varphi' - \nu\varphi]_{r=1} = 0 \tag{3.21}$$

Multiplying (3.21) by $\varepsilon^\alpha \eta$ integrating between 0 and 1, and using (3.18), we have in case 2) in (3.17)

$$\varepsilon^{2+\alpha} \int_0^1 \left(r\varphi'\eta' + \frac{1}{r}\varphi\eta \right) dr - \nu\varepsilon^{2+\alpha}\eta(1)\varphi(1) - \varepsilon^\alpha \int_0^1 \theta\eta^2\, dr -$$
$$\varepsilon^{1+\alpha} \int_0^1 (s_4 + \varepsilon u^\circ)\eta^2\, dr = \varepsilon^\alpha \int_0^1 (u_* - u_\varepsilon)\eta^2\, dr \tag{3.22}$$

Using the obvious inequalities

$$-\theta - \varepsilon(s_4 + \varepsilon u^\circ) \geqslant {}^1/_2\beta r \geqslant 0,\quad {}^1/_8\varepsilon^2 > \varepsilon^{2+\alpha}$$
$$\int_0^1 \left(r\varphi'\eta' + \frac{1}{r}\varphi\eta \right) dr \leqslant \frac{1}{2}\| \varphi \|^2 + \frac{1}{2}\| \eta \|^2$$

as well as (3.17) and (3.22), we obtain from (3.20)

$$\delta^2 I_*(u_*) \geqslant \frac{3}{8}\varepsilon^2 (\| \eta \|^2 + \| \varphi \|^2) - \frac{1}{4}\nu\varepsilon^{2+\alpha}\eta(1)\varphi(1) + \tag{3.23}$$

$$\frac{\nu}{2}\varepsilon^2[\eta^2(1)-\varphi^2(1)]+\frac{1}{8}\beta\varepsilon^{\alpha}\int_0^1 r\eta^2\,dr+\frac{\varepsilon}{2}\int_0^1(s_3+\varepsilon v^{\circ})\eta^2\,dr+$$
$$\frac{1}{2}\int_0^1(v_*-v_\varepsilon)\eta^2\,dr-\frac{1}{4}\varepsilon^{\alpha}\int_0^1(u_*-u_\varepsilon)\eta^2\,dr$$

Setting $0<\alpha<1$ and using the inequalities

$$\beta>0,\ \varepsilon^n\ll\varepsilon^{\alpha+1},\ \nu\varphi^2(1)\leqslant{}^1/_2\|\varphi\|_1^2\quad {}^1/_4\beta\varepsilon^{\alpha}r-\varepsilon(s_3+\varepsilon v^{\circ})\geqslant{}^1/_8\beta\varepsilon^{\alpha}r$$

and the estimate for V_s-V_ε, we derive the positive-definiteness of the second variation from (3.23)

$$\delta^2 I_s(u_*)\geqslant\varkappa/16$$
$$\varkappa=\varepsilon^2(\|\eta\|_1^2+\|\varphi\|_1^2)+\beta\varepsilon^{\alpha}\int_0^1 r\eta^2\,dr>0$$

Now the stability of V_s in the class of axisymmetric functions for problem (3.16) and 2) in (3.17) results from /16/.

In the case of the boundary conditions 1) in (3.17) we obtain (3.22) but with $\nu\varepsilon^{2+\alpha}\eta(1)\varphi(1)$ replacing $\varepsilon^{2+\alpha}\eta(1)\varphi'(1)$. To estimate this component, we multiply (3.21) by $r\varphi'(r)$, integrate between 0 and 1 taking the condition $\varphi(1)=0$ into account, and apply the Cauchy inequality. We consequently obtain

$$\varepsilon^2|\varphi'(1)|^2\leqslant 2\left|\int_0^1[\theta+\varepsilon(s_4+\varepsilon u^{\circ})]\eta\varphi' r\,dr\right|+2\left|\int_0^1(u_\varepsilon-u_*)\eta\varphi' r\,dr\right|\leqslant \tag{3.24}$$
$$m_6^2\|r^{1/2}\eta\|\|r^{1/2}\varphi'\|$$
$$\varepsilon^{2+\alpha}|\eta(1)\varphi'(1)|\leqslant{}^1/_2\varepsilon^{1+\alpha}[\eta^2(1)+\varepsilon^2\varphi'^2(1)]\leqslant$$
$$m_7\varepsilon^{1+\alpha}K_1,\ K_1=\eta^2(1)+\|r^{1/2}\eta\|^2+\|r^{1/2}\varphi'\|^2$$

In the same as when deriving (3.23) by applying the inequality (4.31) in /13/ for $u=\eta$, formulas (3.24) for $1<\alpha<2, 0<\delta<2-\alpha$ and the estimate for V_s-V_ε in the case of sufficiently small ε, we obtain from (3.20) for $\varphi(1)=0$

$$\delta^2 I_s(u_*)\geqslant\frac{\varkappa}{4}-\frac{1}{2}m_7\varepsilon^{1+\alpha}K_1+\frac{1}{2}\int_0^1\{\varepsilon(s_3+\varepsilon v^{\circ})+$$
$$(v_*-v_\varepsilon)-\varepsilon^{\alpha}(u_*-u_\varepsilon)\}\eta^2\,dr\geqslant\varkappa/8>0$$

The stability of V_S in the class of axisymmetric functions therefore results for problem (3.16) and 1) in (3.17).

Stability of the zero-th equilibrium for all $\varepsilon>0$ is obtained from (3.20) for $u_*=v_*=0$. The existence of a saddle-point of the functional I_s is established by using Theorems 2.1 and 2.2 in /6/ and the fact proved earlier /6, 10/ that I_s is representable in the form (1.1) since the last integral in (3.19) is a weakly continuous functional in H_1.

4. DS of spherical and conical shells. The problem of the DS of shallow spherical and conical shells subjected to a uniformly distributed external pressure pulse is written in dimensionless variables in the form

$$w_{\tau\tau}+I_0'(w)=0,\ w(0)=0,\ w_\tau(0)=4\alpha \tag{4.1}$$

$$I_0(w,p)=\frac{1}{2}\|U\|^2+\frac{1}{2}\int_0^{\Lambda}\left(x\Phi_x^2+\frac{1}{x}\Phi^2-2\nu\Phi\Phi_x\right)dx+$$
$$\int_0^{\Lambda}2px^2U\,dx,\quad U=w_x,\quad I_0(w)=I_0(w,p)|_{p=0}$$
$$\|w\|_{H_A}^2=\|U\|^2=\int_0^{\Lambda}\left[xU_x^2+\frac{1}{x}U^2+2\nu UU_x\right]dx$$
$$w|_{x=\Lambda}=[U_x+\nu U/x]_{x=\Lambda}=0,\ \nu={}^1/_3$$
$$1)\ \ \Phi=x\int_x^{\Lambda}y^{-3}\,dy\int_0^y\eta\left(\frac{1}{2}U^2-\theta U\right)d\eta \tag{4.2}$$
$$2)\ \ \Phi=x^{-1}T(x)+\Lambda^{-2}(1+\nu)(1-\nu)^{-1}T(\Lambda)x$$

$$T(x) = \int_0^x y\,dy \int_0^\Lambda \eta^{-1}\left(\frac{1}{2}U^2 - \theta U\right)d\eta$$

Here H is a Hilbert space of the functions in $[0, \Lambda]$, square summable with weight x, and H_A is the energy space of the operator $A = \nabla^4$. The dimensionless and dimensional quantities are related by the formulas

$$\Lambda^2 = 4\,[3\,(1-\nu^2)]^{1/4} h_0 h^{-1},\ 2wh_0 = \Lambda^2 W$$
$$a^2\tau = 2h_0\,(E/m)^{1/2} t,\ W|_{t=0} = 0,\ W_t|_{t=0} = b$$
$$3\,(1-\nu^2)\Lambda^2 a^2 I = 2\pi h_0^3 h^3 E I_0,\ ax = r\Lambda$$
$$16\alpha = b\,(\Lambda a/h_0)^2 (m/E)^{1/2},\ q_0 p = X^*,$$
$$q_0 = 32 E h_0^3 h \Lambda^{-8} a^{-4},\ b = \text{const}$$

Here $W(r, t)$ is the deflection at the time t at a point with polar coordinate r, a is the radius of the reference contour, h is the thickness, m is the shell mass per unit volume, E is Young's modulus, and I is the shell potential energy under the uniform external pressure X^*. For a spherical shell $x— = \theta$ and h_0 is the rise, while for a conical shell $\theta_* = -\Lambda$ and h_0 is half the altitude at the apex. The stress function Φ is written in the form 1) and 2) in (4.2), respectively, in the case of a moving and fixed hinge support.

It follows from numerical calculations of the stationary problem by using the alignment and matrix factorization methods /17/ for closed framing and free clamping of the edge for $\Lambda \leqslant 20$, that stiffness of the spherical shell occurs /19/. For a moving hinge support and fixed hinge support of the edge for $\Lambda \geqslant 5.535$ and $\Lambda \geqslant 3.25$, respectively, the spherical shell has at least one non-zero saddle equilibrium in the absence of a load.

Figs.1-5 show graphs of the potential energy of spherical and conical shells under a uniform external pressure, where $E_0 = 10^{-2} I_0\,(w^*, p)$, and $I_0'\,(w^*, p) = 0$.

The graphs in Fig.1 refer to a spherical shell with moving hinge support: the dashed line is for $\Lambda = 5.75$ in the right upper corner, and the solid line in the centre for $\Lambda = 10$. When there is no load ($p = 0$) there are two non-trivial solutions for $\Lambda = 5.75$ and ten for $\Lambda = 10$. Note that part of the graph for $\Lambda = 10$ has been represented earlier in /18/. In the case of a spherical shell with fixed hinge support we obtain two and six non-trivial solutions, respectively, together with the zero-th solution for $p = 0$ from analogous graphs in Fig.2 represented by dashed lines for $\Lambda = 3.25$ and solid lines for $\Lambda = 5$.

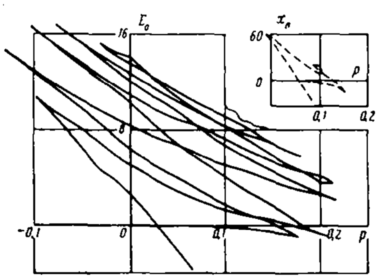

Fig.1

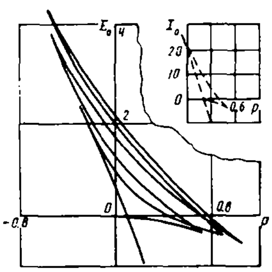

Fig.2

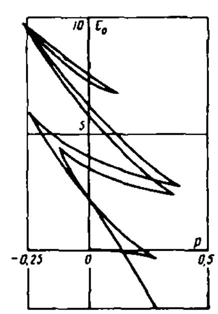

Fig.3

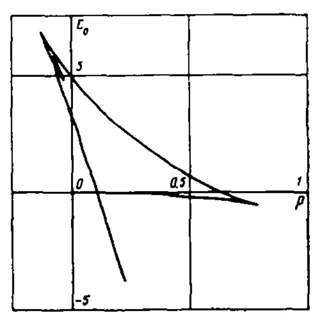

Fig.4

Numerical computations for a shallow conical shell show that loss of stiffness occurs for $\Lambda \geqslant \Lambda_1^* = 5.25$; $\Lambda \geqslant \Lambda_2^* = 2.75$, $\Lambda \geqslant \Lambda_3^* = 6$ respectively, for the moving hinge support, fixed hinge support, and free clamping of the edge. No non-trivial solutions are found for closed framing of the edge for $p = 0$. Graphs of the potential energy are represented for conical shells in Fig.3 for the case of a moving hinge support with $\Lambda = 6$ and in Fig.4 for a fixed hinge support with $\Lambda = 7$, and in Fig.5 for free clamping of the edge with $\Lambda = 8$.

We will now calculate the astatic critical impulse α_a. To determine it from (2.2) in the case (4.1) we obtain the system

$$I_0(w_a) = 4\alpha^2\Lambda^2, \; I_0'(w_a) = 0, \; I(0) = 0 \tag{4.3}$$

where w_a is unstable equilibrium. Let B_α be a set of solutions of system (4.3) arranged in the form of a non-decreasing sequence of numbers α_i $(i = 1, 2, \ldots, n)$. It is evident that $\alpha_a \in B_\alpha$ and $\alpha_1 \leqslant \alpha_a$. If the set B_α consists of one point, then $\alpha_a = \alpha_1$. If $i > 1$, then a difficulty arises in extracting those saddles that belong to the boundary of the well $\partial J(0)$. Only the solution of the non-stationary problem (4.1) and (4.2) yields a guarantee that the saddle would belong to $\partial J(0)$.

Below we show the two least values from the set B_α for a spherical shell with a moving and fixed hinge support of the edge, respectively

Λ	5.535	5.75	6.75	8	9	10
$100\alpha_1$	61.8	63.8	74.5	88.3	99	109
$100\alpha_2$	—	—	88.4	105	116	126
$100\alpha_d$	73	76	95	111	122	132
Λ	3.25	3.94	5	6	7	8
$1000\alpha_1$	69.3	83	99.3	114	127	139
$100\alpha_2$	—	96.7	116	131	143	155
$100\alpha_d$	77.3	—	132	150	169	182

Values of Λ are given for which B_α consists of one element $\alpha_a = \alpha_1$. In the remaining cases we obtain that $\alpha_a = \alpha_2$, where α_2 is on the branch of the unstable solutions having a common point with the stable precritical equilibriums for p equal to the upper critical load p_u.

The critical DS pulse α_d was evaluated by direct numerical integration of problems (4.1) and (4.2) for different α by using an implicit finite-difference scheme and the Budyansky-Roth criterion.

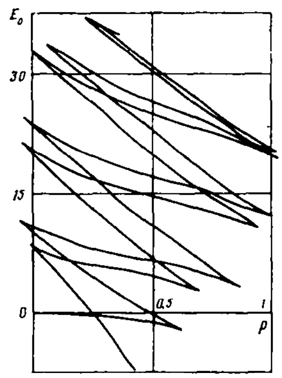

Fig.5

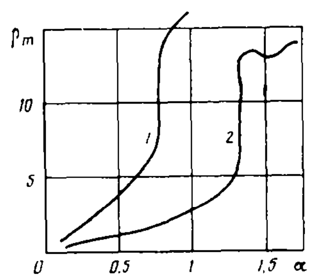

Fig.6

Formulation of this criterion for this case reduces to the following. Let

$$\rho(\tau) = \frac{2}{\Lambda^2}\int_0^\Lambda w(x,\tau)\,x\,dx, \quad \rho_m = \max_{0\leqslant\tau\leqslant T} |\rho(\tau)|$$

Its curve $\rho(\tau)$ and the point $\rho_m(\alpha)$ correspond to each value of α. The critical impulse α_d of a shell axisymmetric DS is defined as the least value of α for which the curve $\rho_m(\alpha)$ has a jumplike change. This method was used earlier /19/ in order to show that there is no DS for spherical shells with closed framing of the edge subjected to a PL. In the calculations it was assumed that $T = 500$

Fig.6 shows graphs of $\rho_m(\alpha)$ for a moving hinge support with $\Lambda = 5.75$ (curve *1*) and a fixed hinge support with $\Lambda = 5$ (curve *2*). The corresponding values of α_d are represented above.

We will give some examples of values calculated by (4.3) for a conical shell for the above-mentioned boundary conditions and values of Λ_i^*: $\alpha_a(\Lambda_1^*) = 1.197$; $\alpha_a(\Lambda_2^*) = 0.987$; $\alpha_a(\Lambda_3^*) =$

1.69.

5. Initially loaded shells. Let $I = I(\omega, p)$ depend on the load parameter p and let $\omega_s(p)$ be the family of stable equilibria of the system (1.1), $p \in [0, p_u)$, where p_u is the upper critical load of static buckling. The system is first subjected to quasistatic loading and is consequently in equilibrium $\omega_0 = \omega_s(p_0)$ corresponding to the value p_0. Then, an additional PL is applied to the system, and is reduced until the system starts to move, and therefore communicates the velocity v_0 to it. In this case the astatic critical PL α_a of the equilibrium ω_0 is determined as the least solution of the equation

$$I(\omega_a, p_0) = \frac{1}{2}\|v_0(\alpha)\|_H^2 + I(\omega_0, p_0), \quad I'(\omega_a, p_0) = 0 \tag{5.1}$$

where ω_a is unstable equilibrium on the boundary of the well $\partial J(\omega_0)$.

Obviously, the DS of a system subjected to a PL is possible if $p_0 \geqslant p_l$, where p_l is the lower critical load of static buckling.

As an example, we consider a spherical shell with closed framing of the edge for $\Lambda = 5$, which is in equilibrium under the action of a hydrostatic load corresponding to $p_0 = 0.30$. At the time $t = 0$ a uniformly distributed external PL that communicates a velocity $W_t|_{t=0} = b$ to the shell, is applied to the surface. We obtain as a result of calculations by means of (4.1) that $I_0(w_0, p_0) = -2.42$ and $I_0(w_a, p_0) = 10.59$. We hence find $\alpha_a = 0.36$ by using (5.1).

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REFERENCES

1. HOFF N.J. and BRUCE V.G., Dynamic analysis of the buckling of laterally loaded flat arches, J. Math. and Phys., 32, 4, 1953.
2. HSU C.S., On dynamic stability of elastic bodies with prescribed initial conditions, Intern. J. Eng. Sci., 4, 1, 1966.
3. SRUBSHCHIK L.S., On the critical pressure of dynamic snap-through of elastic spherical and conical shells. Dep. 1506-83. VINITI, Rostov on Don, 1983.
4. SRUBSHCHIK L.S., Energy criterion of the dynamic snap-through of elastic spherical shells, Dokl. Akad. Nauk SSSR, 280, 1, 1985.
5. SRUBSHCHIK L.S., Non-linear analysis of the axisymmetric dynamic snap-through of elastic spherical shells. Proceedings of the First All-Union Symposium on "Non-linear Theory of Thin-walled Structures and Biomechanics", Izd. Tbilisi Univ., 1985.
6. SRUBSHCHIK L.S. and YUDOVICH V.I., On the dynamic snap-through of a non-linear elastic system, PMM, 50, 3, 1986.
7. MYSHKIS A.D., Troughs in topological spaces, Matem. Zametk. 33, 2, 1983.
8. SHU S.S., Parametric excitation and snap-through problems for shells. Thin-walled Shell Structures, Mashinostroyenie, Moscow, 1980.
9. SIMITSES G.J., Suddenly-loaded structural configurations, J. Eng. Mech., 110, 9, 1984.
10. VOROVICH I.I., On certain direct methods in non-linear theory of shallow shell vibrations, Izv. Akad. Nauk SSSR, Ser. Matem., 21, 6, 1957.
11. VOROVICH I.I., Non-uniqueness and stability in the non-linear mechanics of a continuous medium. Certain mathematical problems. Unsolved Problems of Mechanics and Applied Mathematics. Izd. Mosk. Gos. Univ., 1977.
12. SRUBSHCHIK L.S., Non-stiffness of a spherical shell, PMM, 31, 4, 1967.
13. SRUBSHCHIK L.S., On the question of non-stiffness in the non-linear theory of shallow shells, Izv. Akad. Nauk SSSR, Ser. Matem., 36, 4, 1972.
14. VISHIK M.I. and LYUSTERNIK L.A., Regular degeneration and boundary layer for linear differential equations with a small parameter, Usp. Matem. Nauk, 12, 5, 1957.
15. VASIL'YEVA A.B. and BUTUZOV V.F., Asymptotic Expansions of Solutions of Singularly Perturbed Equations, Nauka, Moscow, 1973.
16. SRUBSHCHIK L.S. and YUDOVICH V.I., Remark on the stability of membrane solutions in the non-linear theory of plates and shells, PMM, 30, 1, 1966.
17. SRUBSHCHIK L.S., Non-stiffness of thin strictly convex shells under fixed hinge support of an edge, Dokl. Akad. Nauk SSSR, 262, 1982.
18. VALISHVILI N.V., Methods of Analysing Shells of Revolution on Digital Computers. Mashinostroyenie, Moscow, 1976.
19. HUANG N.C., Axisymmetric dynamic snap-through of elastic clamped shallow spherical shells. AIAA Jnl, 7, 2, 1969.

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